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Chaotic Properties of Multipoint Correlation Functions of an Ising Model with Long-Range Interactions on the Sierpiński–Gasket Lattice

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A class of multispin correlation functions of an Ising model with ferromagnetic nearest neighbor interactions K and constant (distance-independent) long-range interactions $Q_l = Q$, $l = 1, 2, \ldots$, on the Sierpiński-gasket lattice is considered. Using an exact method for calculating thermodynamic functions of hierarchically constructed Ising systems, it is shown that, for a set of values of Q and for almost all values of K, some M_k-spin correlation functions, where $M_k = 3^k + 3$ with $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ being the order of lattice construction, change chaotically as n, k, and thereby M_k increase to infinity. Accordingly, in the thermodynamic limit, these correlation functions prove to be nonanalytic for appropriate values of Q and K. Since M_k-point correlation functions with k being finite, i.e., correlation functions involving finite numbers of spins, remain analytic as n tends to infinity, there is a smooth crossover between analytic properties of correlation functions of the two types.

KEY WORDS: Ising model with long-range interactions; Sierpiński-gasket lattice; correlation functions; chaos.

1. INTRODUCTION

Using recursive methods to study statistical systems, one can reduce the analysis of their thermodynamic properties to investigating features of appropriate discrete maps. Perhaps the best-known example of such a reduction is the application of the renormalization-group (RG) approach to a given model and the generation of the RG flow in the space of couplings. Usually, the maps characterizing properties of statistical systems display a simple asymptotic behavior, i.e., they possess stable and/or

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unstable fixed points.[1] There are, however, real systems described by maps which exhibit more complex asymptotic behaviors, such as stable and unstable limit cycles of different orders, or even chaotic behavior. Indeed, complex properties of one-dimensional and multidimensional RG trajectories have been found in cases of hierarchical spin systems[2–5] and a two-dimensional Lie-group-valued spin model,[6,7] respectively. It turns out that various nontrivial behaviors of maps characterizing statistical systems result in specific properties of thermodynamic functions of these systems. For instance, the chaotic RG flow of a given statistical model is associated with a nonanalycity of the free energy over a range of temperatures.[4] In the case of hierarchical Ising models, the occurrence of chaotic RG flows has been interpreted as a consequence of the existence of a spin-glass phase[2,5] or incommensurate structures.[3]

In this paper, a class of multipoint correlation functions of an Ising model with positive nearest neighbor couplings K and negative long-range interactions Q_l, $l = 1, 2, ...$, on the Sierpiński-gasket (SG) lattice is considered. The long-range interactions are introduced in a self-similar manner, reflecting the hierarchical structure of the lattice,[8] and are assumed to be all equal to a constant Q (independent of the distance on which they act). It should be noted that, although multipoint correlation functions of translationally invariant systems have been studied for many years, both in the critical region and away from this region,[10–13] the multipoint correlation functions of fractal lattice models have not been analyzed as yet. Since fractal lattices are inhomogeneous at all scales, correlation functions of spin models defined on such lattices depend, in general, not exactly on distances between given spins, but on their positions as well.[8,14] Consequently, in order to study some global (orientationally independent) properties of spin correlations of fractal lattice systems in the critical region, one ought to resort to effective correlation functions, averaged geometrically over positions of all spins involved by these functions.[8] On the other hand, certain global features of spin correlations of fractal lattice systems can be investigated by considering multipoint correlation functions, which involve spins distributed in the same way among some vertices of each of identical lattice fragments being parts of the self-similar structure of a given lattice. The analysis of properties of such multipoint correlation functions yields an insight into the nature of the competition between short-range and long-range interactions.

The correlation functions studied here are formed from spins located at corners of the largest upward-pointing triangle (created at the zeroth level of the lattice construction) and at corners of all downward-oriented triangles, created at the kth, intermediate, stage of construction of the gasket, with $k = 1, 2, ..., n$, $n \geqslant 1$, where n denotes the order of the construction of the ultimate SG lattice. At the nth level of construction of the system (for the method of hierarchical creation of SG-like lattices, see ref. 15), the number of such spins is equal to $M_k = 3^k + 3$, while the total number of lattice points amounts to $N_n = (3^{n+1} + 3)/2$, $n \geqslant 0$. Below, it is shown that, for some nonzero finite temperatures and for some values of Q belonging to a finite interval, the multipoint correlation functions under consideration here do not tend to a limit as n and k $(k \leqslant n)$ increase to infinity. In particular, for some isolated points from this interval and for almost all values of K, the multispin correlation functions exhibit chaotic behavior as n and k grow.

2. THE LONG-RANGE INTERACTING FRACTAL MODEL

The SG-lattice spin model with long-range interactions $Q_l \geqslant 0$, $l = 1, 2, ...,$ distributed in a self-similar way along edges of all upward-pointing triangles of all scales, excepting the smallest upward-oriented triangles (see Fig. 1), has turned out to be very useful for studying critical phenomena in fractal systems.[8,16] Due to the occurrence of the long-range couplings, the system is infinitely ramified and reveals phase transitions at nonzero finite temperatures, both for constant Q_l, i.e., for distance-independent long-range interactions,[8] and for Q_l decaying algebraically with the distance.[9] It is remarkable that, in the case of constant long-range interactions, the model is exactly tractable.[8,9] Thus, in this case, the system provides a useful testing model for studying in an exact way various general questions concerning phase transitions in fractal spin systems at nonzero

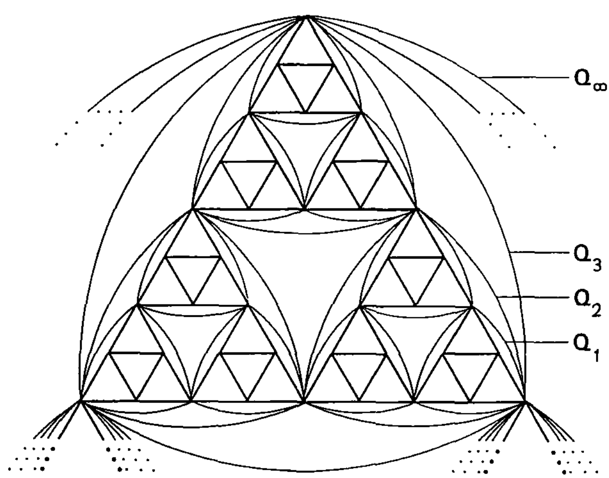

Fig. 1. The SG lattice. Long-range interactions Q_l are also represented.

temperatures. In particular, investigations of thermodynamic properties of this system have shown that a classification of fractal spin systems according to universality is possible, and have shown how to construct appropriate universality classes.[16]

Here, a system of Ising spins $\sigma = \pm 1$ on the SG lattice with ferromagnetic nearest neighbor interactions K and with antiferromagnetic constant long-range interactions $Q_l = Q \leqslant 0$, $l = 1, 2, \ldots$, is considered. The Hamiltonian of the system generated at the nth, $n \geqslant 1$, level of construction of the gasket can be taken in the form

$$\mathscr{H}(\sigma)/k_B T = \sum_{\mathbf{R}_{n-1}}^{\nabla} \sum_{\alpha,\beta=1}^{3} K_\alpha(\mathbf{R}_{n-1} + \mathbf{e}_\beta)\, \sigma(\mathbf{R}_{n-1} + \boldsymbol{\rho}_{\alpha,\beta})\, \sigma(\mathbf{R}_{n-1} + \boldsymbol{\rho}_{\alpha+1,\beta})$$
$$+ \sum_{l=1}^{n} Q_l \sum_{\mathbf{R}_{n-1}}^{\nabla} \sum_{\alpha=1}^{3} \sigma(\mathbf{R}_{n-1} + 2^l \mathbf{e}_\alpha)\, \sigma(\mathbf{R}_{n-1} + 2^l \mathbf{e}_{\alpha+1}) \qquad (2.1)$$

with $K_\alpha \equiv K$, $\alpha = 1, 2, 3$, being the nearest neighbor couplings. Such a notation is introduced to record the spatial orientation of the couplings K along edges of each of the smallest upward-pointing triangle (see Fig. 2a), and has no intrinsic significance. The long-range interactions $Q_l = Q \leqslant 0$ are assumed to act at the distances 2^l, $l = 1, 2, \ldots$ (measured in units of the lattice constant). The symbol $\sum^{\nabla}$ means that the vectors $\mathbf{R}_m \equiv (X, Y)_m$, $m = 0, 1, \ldots$, run through the centers of all downward-oriented triangles

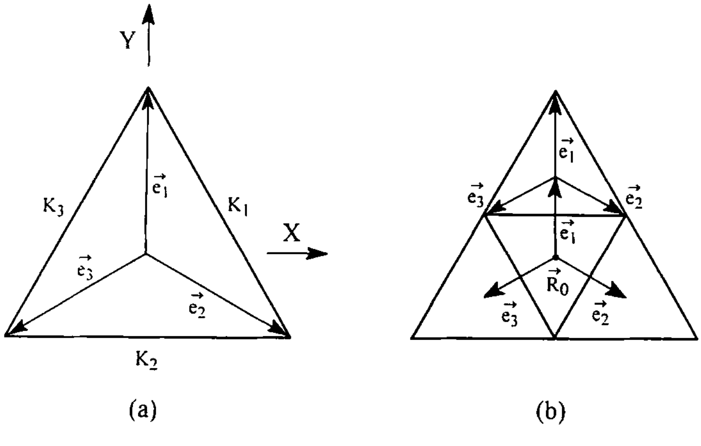

Fig. 2. (a) Spatial orientation of the interactions K_α, $\alpha = 1, 2, 3$, in the basic (smallest) upward-pointing triangle of the SG lattice. Orientation of the vectors $\mathbf{e}_\alpha$, $\alpha = 1, 2, 3$, introduced in the text, is also shown. (b) The SG lattice at the first construction level. The manner of labeling lattice points is illustrated.

generated at the mth stage of construction of the SG lattice. The basic vectors $\mathbf{e}_\alpha$, $\alpha = 1, 2, 3$, and $\mathbf{e}_4$ are defined by (see Fig. 2a)

$$\mathbf{e}_1 = (0, \tfrac{1}{3}\sqrt{3}) \tag{2.2}$$

$$\mathbf{e}_2 = (\tfrac{1}{2}, -\tfrac{1}{6}\sqrt{3}) \tag{2.3}$$

$$\mathbf{e}_3 = (-\tfrac{1}{2}, -\tfrac{1}{6}\sqrt{3}) \tag{2.4}$$

$$\mathbf{e}_4 \equiv \mathbf{e}_1 \tag{2.5}$$

while the vectors $\boldsymbol{\rho}_{\alpha,\beta}$ and $\boldsymbol{\rho}_{4,\beta}$, $\alpha, \beta = 1, 2, 3$, are determined by

$$\boldsymbol{\rho}_{\alpha,\beta} = \mathbf{e}_\alpha + \mathbf{e}_\beta \tag{2.6}$$

$$\boldsymbol{\rho}_{4,\beta} \equiv \boldsymbol{\rho}_{1,\beta} \tag{2.7}$$

Thus, the vectors $\mathbf{R}_{n-1} + \mathbf{e}_\beta$, $\beta = 1, 2, 3$, indicate the locations of the centers of the three smallest upward-pointing triangles generated at the nth stage of the lattice construction and contained in a larger upward-pointing triangle (of linear size 2) created at the $(n-1)$th lattice construction level and centered at $\mathbf{R}_{n-1}$. Accordingly, the vectors $\mathbf{R}_{n-1} + \boldsymbol{\rho}_{\alpha,\beta}$ and $\mathbf{R}_{n-1} + \boldsymbol{\rho}_{\alpha+1,\beta}$, $\alpha, \beta = 1, 2, 3$, indicate positions of vertices of these three smallest triangles. (Note that, for convenience, the lattice constant is assumed here to be equal to one at each construction level of the lattice.) Consequently, the vectors $\mathbf{R}_{n-1} + 2^l\mathbf{e}_\alpha$ and $\mathbf{R}_{n-l} + 2^l\mathbf{e}_{\alpha+1}$, $\alpha = 1, 2, 3$, indicate positions of vertices of a downward-pointing triangle of linear size 2^l, generated at the (n-l)th lattice construction stage and centered at $\mathbf{R}_{n-l}$ (see Fig. 2b).

3. GENERATING EQUATIONS FOR MULTIPOINT CORRELATION FUNCTIONS

Consider, for the model defined in Eqs. (2.1)–(2.7), a class of M_k-point ($M_k = 3^k + 3$) correlation functions $\Gamma_{n,k}(\{K_\alpha\}, Q)$, $k = 1, 2, ..., n$, $n \geqslant 1$, involving spins located at corners of the largest (upward-pointing) triangle and at corners of all downward-oriented triangles generated at the kth stage of construction of the SG lattice. Thus, the M_k spins involved in these correlation functions are assumed to be distributed in the same manner for each n and k, among appropriate vertices of each of identical lattice fragments being parts of the self-similar structure of the gasket. Such a distribution of spins adopted in the definition of $\Gamma_{n,k}(\{K_\alpha\}, Q)$ reflects the hierarchical structure of the SG lattice. In the case of $k=n$, $n \geqslant 1$, the correlation functions can be determined by

$$\Gamma_{n,n}(\{K_\alpha\}, Q) = \prod_{\mathbf{R}_{n-1}}^{\nabla} \prod_{\alpha=1}^{3} \frac{d}{dK_\alpha(\mathbf{R}_{n-1}+\mathbf{e}_\alpha)} Z_n(\{K_\alpha\}, Q) \tag{3.1}$$

where $\prod^{\nabla}$ denotes that $\mathbf{R}_{n-1}$ runs through centers of all downward-pointing triangles generated at the $(n-1)$th, $n \geqslant 1$, stage of lattice construction, and Z_n is the partition function of the nth, $n \geqslant 1$, level system. For each $k \leqslant n$, $\Gamma_{n,k}$ can also be determined by multiple differentiation of the partition function with respect to some nearest neighbor interactions of appropriate orientations and associated with the smallest upward-pointing triangles forming the SG lattice.

The partition function of the system at each level of the gasket construction can be determined using an exact recursive method[9,15] for calculating the constrained partition function $Z_i^{\sigma_1\sigma_2\sigma_3}$ with arbitrary configuration $(\sigma_1, \sigma_2, \sigma_3)$ of spins located at corners of the largest triangle of the SG lattice (generated at the ith construction stage). Then, one has

$$\begin{aligned} Z_{i+1}^{\sigma_1\sigma_2\sigma_3}(u, v) = {} & Z_i^{\sigma_1\sigma'\sigma''}(u, v)\, Z_i^{\sigma_2\sigma''\sigma'''}(u, v)\, Z_i^{\sigma_3\sigma'''\sigma'}(u, v) \\ & \times w(1+vS_{123}) \end{aligned} \tag{3.2}$$

where the convention of summation over repeated indices $(\sigma', \sigma'', \sigma''')$ is applied, $S_{123} = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3$, and

$$u = 1/t_K - 1 + t_K \tag{3.3}$$

$$v = 1/(1/t_Q - 1 + t_Q) \tag{3.4}$$

$$w = c_Q^3(1 + t_Q^3) \tag{3.5}$$

with $t_x = \tanh(x)$ and $c_x \equiv \cosh(x)$. Note that the constrained partition function depends, in principle, on K (or K_α, $\alpha = 1, 2, 3$) and Q. The variables u and v have been introduced for the sake of simplicity. Using (3.2), one can calculate the constrained partition function for increasing i, beginning with

$$Z_0^{\sigma_1\sigma_2\sigma_3}(u, v) \equiv Z_0^{\sigma_1\sigma_2\sigma_3}(u) = A_0(u_0 + S_{123}) \tag{3.6}$$

where

$$A_0 = c_K^3 t_K(1 + t_K) \tag{3.7}$$

$$u_0 \equiv u \tag{3.8}$$

Then, at the $(i+1)$th iteration, one derives

$$Z_{i+1}^{\sigma_1\sigma_2\sigma_3}(u, v) = A_{i+1}(u_{i+1} + S_{123}) \tag{3.9}$$

with

$$A_{i+1} = 8A_i^3 w(1+u_i)[(1-u_i+u_i^2)v + 1 + 2v] \tag{3.10}$$

$$u_{i+1} = (1-u_i+u_i^2+3v)/[(1-u_i+u_i^2)v+1+2v] \equiv g(v, u_i) \tag{3.11}$$

Hence, the partition function

$$Z_{i+1}(u, v) = \sum_{\sigma_1, \sigma_2, \sigma_3} Z_{i+1}^{\sigma_1\sigma_2\sigma_3}(u, v)$$

is given by

$$Z_{i+1}(u, v) = 8A_{i+1}u_{i+1} \tag{3.12}$$

Note that the processes of construction of the SG lattice and the calculation of the partition function proceed, in some sense, in opposite directions. The first process relies on discarding from the SG lattice downward-pointing triangles of smaller scale than the scale of the smallest triangles forming the lattice at a given construction stage,[15] whereas the second process is associated with creating a larger lattice from identical smaller fragments.[17]

It is obvious that the recursive method of hierarchical calculation of the partition function can be adopted to determine the correlation functions $\Gamma_{n,k}$. Indeed, these functions can be expressed as

$$\Gamma_{n,k}(u, v) = [Z_n(u, v)]^{-1} \sum_{\sigma_1, \sigma_2, \sigma_3} \gamma_{n-k,k}^{\sigma_1\sigma_2\sigma_3}(u, v) \tag{3.13}$$

with $\gamma_{n-k,k}^{\sigma_1\sigma_2\sigma_3}$ being determined by the iteration relation

$$\gamma_{n-k,i+1}^{\sigma_1\sigma_2\sigma_3}(u, v) = \gamma_{n-k,i}^{\sigma_1\sigma'\sigma''}(u, v)\, \gamma_{n-k,i}^{\sigma_2\sigma''\sigma'''}(u, v)\, \gamma_{n-k,i}^{\sigma_3\sigma'''\sigma'}(u, v) \times w(1+vS_{123}), \qquad i = 0, 1,..., k-1 \tag{3.14}$$

starting with

$$\gamma_{n-k,0}^{\sigma_1\sigma_2\sigma_3}(u, v) = Z_{n-k}^{\sigma_1\sigma'\sigma''}(u, v)\, Z_{n-k}^{\sigma_2\sigma''\sigma'''}(u, v)\, Z_{n-k}^{\sigma_3\sigma'''\sigma'}(u, v) \times \sigma'\sigma''\sigma''' w S_{123}^{(3)}(1+vS_{123}) \tag{3.15}$$

where $S_{123}^{(3)} = \sigma_1\sigma_2\sigma_3$. Note that in (3.14) and (3.15) the summation over repeated indices σ', σ'', and σ''' is understood. Using (3.14) and (3.9)–(3.11), one easily finds that

$$\gamma_{n-k,i}^{\sigma_1\sigma_2\sigma_3}(u, v) = A_{n-k+i}(u_{n-k+i} + S_{123}), \qquad i = 1, 2, ..., k \tag{3.16}$$

where A_{n-k+i} and u_{n-k+i} are determined by the recurrence equations (3.10) and (3.11), with the initial conditions $A_{n-k+1} = \bar{A}_{n-k+1}$ and $u_{n-k+1} = \bar{u}_{n-k+1}$, where

$$\bar{A}_{n-k+1} = 16A_{n-k}^3(u'_{n-k}v + 1 + 2v)\, u'_{n-k}w \tag{3.17}$$

$$\bar{u}_{n-k+1} = (u'_{n-k} + 3v)/(u'_{n-k}v + 1 + 2v) \tag{3.18}$$

with

$$u'_{n-k} = 1/u_{n-k} \tag{3.19}$$

Accordingly, in the special case of $k = n$, i.e., when the correlation function involves spins located at corners of all smallest downward-pointing triangles created at the nth lattice construction level, one has

$$\Gamma_{n,n}(u, v) = A_n u_n / Z_n(u, v) \tag{3.20}$$

with A_n and u_n generated by (3.10) and (3.11), beginning with

$$A_1 = 16A^3\left(\frac{v}{u} + 1 + 2v\right)\frac{w}{u} \tag{3.21}$$

$$u_1 = \left(\frac{1}{u} + 3v\right)\bigg/\left(\frac{v}{u} + 1 + 2v\right) \tag{3.22}$$

4. PROPERTIES OF THE MULTISPIN CORRELATION FUNCTIONS

Before studying properties of the correlation functions $\Gamma_{n,k}$, let us discuss some general thermodynamic properties of the long-range interacting spin model on the SG lattice. Clearly, by means of (3.12), (3.10), and (3.11), the thermodynamic behavior of the system is determined by properties of fixed-point solutions $u^* = u_\infty(v)$ to the iterative equation (3.11). This equation has three such solutions, i.e.,

$$u_1^* = 1 \qquad \text{and} \qquad u_\pm^* = \frac{1}{2v} \pm \frac{1}{2v}(1 - 4v - 12v^2)^{1/2}$$

It can easily be checked that for each $Q>0$ belonging to a finite interval the solution u_+^* is nontrivial, i.e., it is unstable and is associated with $K^*=K(Q)$ being a nonzero finite value of K.[9] Accordingly, the spin system on the SG lattice with ferromagnetic nearest neighbor and long-range interactions exhibits phase transitions at nonzero finite temperatures, accompanied by a nonuniversal critical line $K^*=K(Q)$ with continuously varying exponents.[8, 9]

The occurrence of long-range couplings in the system has a smoothing effect on the structure of the SG lattice. It has been shown that as a consequence of such a smoothing, critical properties of the fractal model with $Q>0$ and short-range interacting models defined on abstract translationally invariant lattices interpolated to noninteger dimensionalities reveal some similarities.[18] However, the similarities are distinct only for values of Q from a rather small interval, and, in general, the long-range interacting spin model on the SG lattice displays thermodynamic properties typical for regular fractal systems. This follows from the fact that, although the long-range couplings are distributed on the SG lattice in a different way than the short-range ones, the distributions of interactions of both types reflect the self-similar fractal structure of the lattice. Accordingly, the competition between short-range and long-range interactions does not lead for any K and Q to breaking *translational invariance*. By contrast, the translational symmetry breaking takes place in cases of Ising models on Cayley trees with short-range interactions and infinite-range couplings between all pairs of spins.[19] In such cases, reducing translational invariance due to the competition between interactions of the two types can be interpreted as lowering the spatial dimensionality.

Consider now the case of $Q\leqslant 0$. Since, by virtue of (3.4), $-1/3\leqslant v\leqslant 0$, (3.11) implies that $u_i\leqslant -3$ and $u_i\geqslant 1$, $i=1, 2, ...$, for $K\geqslant 0$ as well as for $K\leqslant 0$. [It should be pointed out that, from (3.3), one obtains $u\geqslant 1$ for $K\geqslant 0$ and $u\leqslant -3$ for $K\leqslant 0$.] However, as can be easily verified, $0\leqslant u_-^*\leqslant 1$. Thus, for $Q\leqslant 0$, this fixed-point solution of Eq. (3.11) does not correspond to any real K, except for the limit value $u_-^*=1$ associated with $Q=0$, $K=\infty$, and thereby it has no physical meaning. With regard to the two remaining fixed-point solutions of (3.11) with $Q\leqslant 0$, they admit real K (note that, although $u_+^*\leqslant -3$, it can involve both $K\geqslant 0$ and $K\leqslant 0$). Stability properties of a fixed-point solution u^* of Eq. (3.11) can be determined by evaluating the derivative

$$\lambda(v, u^*)=\left.\frac{dg(v, u)}{du}\right|_{u=u^*}=\frac{(-1+2u^*)(1-u^*v)}{[1-u^*+(u^*)^2]\,v+1+2v} \tag{4.1}$$

At u_0^* one obtains that $\lambda(v, u_0^*) = (1-v)/(1+3v) > 1$ for $Q < 0$, whereas at u_+^* one derives $\lambda(v, u_+^*) = (1+6v+u_+^* v)/(1-vu_+^*) < 1$ for $Q \leqslant 0$. Thus, Eq. (3.11) possesses for $Q \geqslant 0$ two "physical" fixed-point solutions. The first one, u_0^*, associated with $K = \infty$, is a trivial unstable fixed point, and the second one, u_+^*, corresponding, in general, to finite $K > 0$ or $K < 0$, is a stable fixed point. Consequently, when $Q \leqslant 0$, the system does not reveal phase transitions at any finite $K > 0$ or $K < 0$.

From a simple analysis of the nature of the nonlinear equation (3.11), it is clear that, when $Q \leqslant 0$, we have $u_i \leqslant -3$, $i \geqslant 1$, for the intial value $u_0 \equiv u \leqslant -3$ associated with $K \leqslant 0$, as well as for $u_0 \equiv u \geqslant 1$, connected with $K \geqslant 0$. Thus, for $Q \leqslant 0$ and $K \leqslant 0$ or $K \geqslant 0$, the successive iterates u_i, $i = 1, 2, \ldots$, can take on any real value outside the interval $J_1 = [-3, 1]$. However, in the case of calculations of $\Gamma_{n,k}$, u_i takes on, after the $(n-k)$th iteration of Eq. (3.11), a new value $u'_{n-k} \in [-1/3, 1]$ [see Eqs. (3.18) and (3.19)]. As can easily be checked, if u_{n-k} belongs to the interval $J_2 = [0, 1]$ and if $-1/4 \leqslant v \leqslant 0$, then all successive iterates u_{n-k+i}, $i = 1, 2, \ldots$, also belong to this interval. Consequently, for $-1/4 \leqslant v \leqslant 0$, the interval J_2 is mapped by g onto itself. It turns out that, within the interval J_2, the map g exhibits for $-1/4 \leqslant v \leqslant -0.271\ldots$, i.e., for $Q \in J_Q$, where $J_Q = [-0.402\ldots, -0.318\ldots]$, more complex asymptotic properties than the existence of fixed points. In particular, the mapping displays infinite cascades of period-doubling bifurcations, chaotic behavior, and stability windows.[20] The bifurcation diagram for u_i, $i = 1, 2, \ldots$, with $u_1 = 0.9$ is shown in Fig. 3. The Lyapunov exponent,[20] defined for the map g by

$$\Lambda(v) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln |g'(v, u_i)| \tag{4.2}$$

where the prime denotes the derivative with respect to u_i and u_1 stands for an initial value of u_i, is illustrated for $0 < u_1 < 1$ in Fig. 4 as a function of v. It is seen that, for some isolated values of v, $\Lambda(v)$ is positive. Consequently, the map g behaves chaotically at these values of v.[21] It should be noted that g has within the interval J_2 a single extremum (at $u_i = 1/2$). Therefore, one can expect that the metric properties of the map g are the same as the universal metric properties of typical maps with quadratic maximum and with negative Schwarzian derivative.[20, 22]

For simplicity, the examination of $\Gamma_{n,k}$ is confined below to the case of $K \geqslant 0$ [note that, by virtue of Eqs. (3.19) and (3.3) with $k = n$, $u'_0 = 1/u \in J_2$ if and only if $K \geqslant 0$]. The discussion of features of the nonlinear map g leads to a general conclusion that, contrary to the free energy per spin, the correlation functions $\Gamma_{n,k}$ with $Q \leqslant 0$ can be nonanalytic functions of K (or temperature) as the thermodynamic limit is approached. This

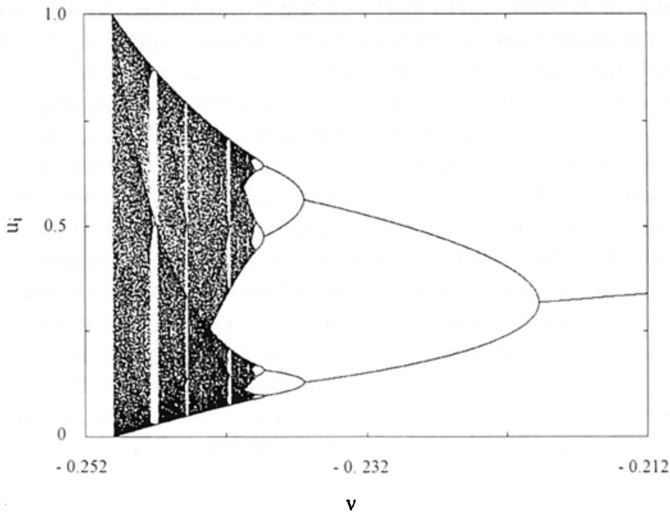

Fig. 3. Bifurcation diagram for the map g. This diagram is generated by taking the initial value for the variable to be $u_1 = 0.9$.

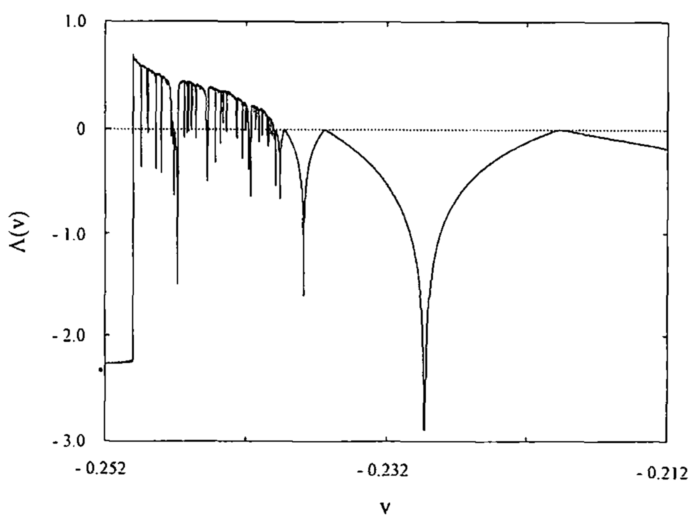

Fig. 4. The Lyapunov exponent as a function of v for the map g with an initial value $u_1 \in J_2$.

can be shown as follows. The partition function and thereby the free energy per spin are determined by (3.11) with the initial value given by (3.8). For this initial condition, $-3 \leqslant u_i$ and $u_i \geqslant 1$, $i = 1, 2, \ldots$, and the partition function per spin is an analytic function of u (and v) at each stage of lattice construction. Although each of the correlation functions $\Gamma_{n,k}$ is also determined by the iteration relation (3.11), it is affected according to (3.19) by a modification of the (n-k)th iterate. As a result of such a modification, the subsequent iterates u_{n-k+i}, $i = 1, 2, \ldots, k$, belong to the interval J_2 if u'_{n-k} belongs to this interval and if $-1/4 \leqslant v \leqslant 0$. Thus, the correlation functions $\Gamma_{n,n-m}$ with $0 \leqslant m < n$ being finite can behave chaotically for each finite m as n grows to infinity, and $\Gamma_{n,n-m}$ are not analytic functions of u (and thereby temperature) in the thermodynamic limit. It must be stressed that a given correlation function $\Gamma_{n,n-m}$ with finite m behaves chaotically as n tends to infinity when Q takes on appropriate values from the interval J_Q and when $u_m > 1$. Obviously, the latter condition is not satisfied for all $K \geqslant 0$. In the case of $m = 0$, $u_0 \geqslant 1$ for all $K \geqslant 0$, and the correlation function $\Gamma_{n,n}$ exhibits chaotic behavior for appropriate values of Q and for almost all values[21] of K. Clearly, in calculating the correlation functions $\Gamma_{n,k}$ for any finite k, the number of iterations of Eq. (3.11) after the modification (3.19) is finite, even if n tends to infinity, and $\Gamma_{n,k}$ with finite k are regular functions of u for all $Q \leqslant 0$. Consequently, there is a smooth crossover between regular and chaotic regimes of the behavior of the correlation functions $\Gamma_{\infty,k}$ as k tends to infinity, i.e., when the number of spins which these correlation functions involve grows to infinity.

Note that when the RG map of a system displays chaotic behavior over a range of temperature, then the system reveals critical phenomena at each temperature from the range, and the free energy of the system is nonanalytic over this temperature range.[4] Since the infinite-range fractal spin model considered here does not reveal critical phenomena at nonzero temperatures, the RG flow of this model is not chaotic. (Notice that the RG equations have the same form[8] in cases of both ferromagnetic and antiferromagnetic long-range interactions.) Accordingly, the free energy of the model is an analytic function of temperature for all nonzero temperatures and for each $Q \leqslant 0$. As is well known, in the presence of external magnetic field H, the free energy can be expressed in a power series in H, with coefficients determined by m-point correlation functions ($m = 1, 2, \ldots$). Obviously, in the case of the model considered here, these coefficients involve correlation functions $\Gamma_{n,k}$, which behave chaotically for some temperatures and for some negative values of Q, as the thermodynamic limit is approached. Thus, one could suppose that the free energy of the system with $H \neq 0$ is a nonanalytic function of temperature (for some values of Q), although for $H = 0$ is an analytic function. However, a given correlation function $\Gamma_{n,k}$ contributes to a coefficient in the expansion of the free energy with the factor $1/M_{n,k}!$. Consequently, in the thermodynamic limit, the chaotic behavior of the functions $\Gamma_{n,k}$ does not influence analytic properties of the free energy for $H \neq 0$.

The influence of negative infinite-range interactions on thermodynamic properties of the spin system on the SG lattice is strongly manifested in the chaotic behavior of the correlation functions $\Gamma_{n,k}$. It follows from Fig. 3 and Eq. (3.4) that the chaotic regime is associated with small values of the coupling parameter Q (i.e., with large absolute values of Q), for which the system is rather highly frustrated. It is also remarkable that, in the chaotic regime, $u_i \geqslant 0$, $i = 1, 2, \ldots$ (see Fig. 3), and, by means of (3.12)–(3.19), the correlation functions $\Gamma_{n,k}$ are ferromagnetic. Thus, although the existence in the system of antiferromagnetic long-range interactions leads to the chaotic behavior of the function $\Gamma_{n,k}$, these interactions are incapable of imposing on $\Gamma_{n,k}$ an antiferromagnetic character.

5. CONCLUDING REMARKS

A class of multipoint correlation functions of the Ising model with constant negative long-range interactions has been considered. The correlation functions involve spins distributed on the lattice in a self-similar manner. These functions can be determined by multiple differentiation of the partition function with respect to the nearest neighbor interactions of appropriate spatial orientations and connected with upward-pointing triangles generated at the last lattice construction stage. In the case when the number of spins involved in the correlation functions tends to infinity as the thermodynamic limit is approached, it has been shown that the correlation functions behave chaotically for some nonzero finite temperatures when the limit of the infinite system is approached, provided that the long-range coupling strength takes on appropriate values from a finite interval. Such a chaotic behavior of the multipoint correlation functions means that in the thermodynamic limit they are nonanalytic functions of temperature. The chaotic behavior of the correlation functions $\Gamma_{n,k}$ is a result of a competition of ferromagnetic short-range and antiferromagnetic long-range interactions. Due to a special distribution of long-range couplings on the SG lattice, the system is frustrated at all length scales, similarly as in Ising models with competing ferromagnetic and antiferromagnetic interactions on hierarchical lattices.[2] However, in contrast with these hierarchical spin models, the system considered in this paper does not exhibit chaotic RG trajectories. It is noteworthy that, contrary to the multispin correlation functions displaying chaos, the free energy per spin is a regular function of the temperature for all negative values of the long-range interaction strength. It should be pointed out that the correlation functions $\Gamma_{n,n-m}$ with any finite m can be expressed in the thermodynamic limit by infinite-order derivatives with respect to K_α, $\alpha = 1, 2, 3$.

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